

# Directed transient long-range transport in a slowly driven Hamiltonian system of interacting particles

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## Abstract

We study the Hamiltonian dynamics of a one-dimensional chain of linearly coupled particles in a spatially periodic potential which is subjected to a time-periodic mono-frequency external field. The average over time and space of the related force vanishes and hence, the system is effectively without bias which excludes any ratchet effect. We pay special attention to the escape of the entire chain when initially all of its units are distributed in a potential well. Moreover for an escaping chain we explore the possibility of the successive generation of a directed flow based on large accelerations. We find that for adiabatic slope-modulations due to the ac-field transient long-range transport dynamics arises whose direction is governed by the initial phase of the modulation. Most strikingly, that for the driven many particle Hamiltonian system directed collective motion is observed provides evidence for the existence of families of transporting invariant tori confining orbits in ballistic channels in the high dimensional phase spaces.

PACS numbers: 05.45.Ac, 05.60.-k, 05.45.Pq, 05.60.Cd

## I. INTRODUCTION

Transport phenomena play a fundamental role in many physical systems. In this context the so called washboard potential due to its ubiquity and simplicity establishes the prototype of a periodic potential being employed in a number of applications including Josephson junctions [1], charge density waves [2], superionic conductors [3], rotation of dipoles in external fields [4], phase-locked loops [5] and diffusion of dimers on surfaces [6, 7, 8, 9, 10, 11, 12] to quote a few. Applying an external time periodic driving to the washboard potential, interesting effects such as phase-locking, hysteresis [13] and stochastic resonance [14] are found. Recent investigations have dealt with the Hamiltonian dynamics of individual particles evolving in a washboard potential whose slope is time-periodically varied by a weak external monochromatic modulation field [15, 16, 17]. Remarkably, for the corresponding one-and-a-half degree of freedom Hamiltonian system it has been demonstrated that adiabatic modulations of the slope of the potential lead to the generation of transient transport dynamics related with enormous directed particle flow. Explanation for this behavior has been given in terms of the underlying phase space structure of the externally driven one degree of freedom system promoting the motion in ballistic channels [18]. Here we consider an extension of previous work on individual particle movement and study the case when the (many) particles interact due to linear couplings constituting a one-dimensional chain. Motion takes place in a phase space for which details of its intricate structures remain elusive not at least due to the high dimensionality. Whether in systems with a large number of (microscopic) degrees of freedom such (macroscopic) behavior as collective motion leading to a directed flow emanates from high dimensional dynamics is not obvious. In systems with at least three degrees of freedom irregular dynamics in the form of Arnold diffusion arises [19]-[22]. This is possible because the KAM tori do not separate the phase space so that the various chaotic regions always overlap and in principle orbits can wander along the entire stochastic layer permeating the whole phase space. Let us recall that in one-and-a-half degree of freedom weakly nonintegrable Hamiltonian systems the motion can stick to the hierarchical structure of persisting tori and islands inside stochastic layers being able to constitute the above mentioned ballistic channels which support directed transport. The question then is whether such trapping regions persist in many degrees of freedom Hamiltonian systems supporting also directed transport. It is the objective of the current work to

demonstrate that even in a system of many coupled particles the generation of a directed flow going along with collective motion is possible indeed.

## II. THE DRIVEN SYSTEM OF INTERACTING PARTICLES

We study a one-dimensional chain system consisting of linearly coupled particles with Hamiltonian of the following form

$$H = \sum_{n=1}^N \left[ \frac{p_n^2}{2} + U_0(q_n) + U_1(q_n, t) \right] + \frac{\kappa}{2} \sum_{n=1}^{N-1} (q_{n+1} - q_n)^2, \quad (1)$$

wherein  $p_n$  and  $q_n$  denote the canonically conjugate momenta and positions of the particles evolving in the periodic, spatially-symmetric (washboard) potential of unit period, i.e.,

$$U_0(q) = U_0(q + 1) = -\cos(2\pi q)/(2\pi). \quad (2)$$

The external, time-dependent forcing field

$$U_1(q, t) = -F \sin(\Omega t + \Theta_0)q \quad (3)$$

causes time-periodic modulations of the slope of the potential. It has to be stressed that there is no bias force involved in the sense that the following average over time and space vanishes, i.e.

$$\int_0^1 dq \int_0^{T=2\pi/\Omega} dt \frac{\partial U(q, t)}{\partial q} = 0, \quad (4)$$

with  $U(q, t) = U_0(q) + U_1(q, t)$ . The particles interact linearly with coupling strength  $\kappa$ . Remarkably, as pointed out in prior literature [15] in the limiting case of uncoupled particles, i.e.  $\kappa = 0$ , there results an (unexpected) asymmetry of the flux of particles, emanating from one potential well, and flowing to the left and right potential wells which indicates the existence of directed transport without breaking the reflection symmetry in space and time in this system. One reason for the occurrence of phase-dependent directed transport is the lowering of the symmetry of the flow in phase space by the ac-field where this asymmetry vanishes only for specific values of the initial phase  $\Theta_0$  [15]. In Ref. [16] the authors report

further on this exceptional situation and show that directed transport is sustained on fairly long time scales despite the presence of chaos. In particular it has been demonstrated that for sufficiently small forcing frequencies,  $\Omega \ll 1$ , the width of the arising chaotic layer diverges leading to a strong enhancement of the chaotic transport [16].

The equations of motion derived from the Hamiltonian in Eqs. (1) read as

$$\frac{d^2 q_n}{dt^2} + \sin(2\pi q_n) = F \sin(\Omega t + \Theta_0) + \kappa[q_{n+1} + q_{n-1} - 2q_n] . \quad (5)$$

The analysis in the current paper deals with zero initial phase of the external force term, viz.  $\Theta_0 = 0$ . The influence of the phase  $\Theta_0$  on the generation of directed motion in the periodic potential  $U_0(q)$  has been studied in [17]. We stress that averaging over the initial phase  $\Theta_0$  yields vanishing net current in the periodically driven system.

### III. THE CASE $N = 2$

Before we embark on the study of systems comprising of many degrees of freedom let us first consider the case of two interacting particles. For a discussion of the dynamics it is convenient to introduce the following canonical change of variables induced by the generating function:  $S = \frac{1}{2}(q_1 + q_2)P_x + \frac{1}{2}(q_1 - q_2)P_y$  relating the old and new variables as

$$p_1 = \frac{1}{2}(P_x + P_y), \quad p_2 = \frac{1}{2}(P_x - P_y), \quad (6)$$

$$Q_x = \frac{1}{2}(q_1 + q_2), \quad Q_y = \frac{1}{2}(q_1 - q_2). \quad (7)$$

The Hamiltonian expressed in the new variables becomes

$$\begin{aligned} H = & \frac{1}{4}(P_x^2 + P_y^2) - \frac{1}{\pi} \cos(2\pi Q_x) \cos(2\pi Q_y) \\ & + 2\kappa Q_y^2 - 2F \sin(\Omega t) Q_x. \end{aligned} \quad (8)$$

The corresponding equations of motion are given by

$$\begin{aligned} \ddot{Q}_x = & -\cos(2\pi Q_y) \sin(2\pi Q_x) \\ & + F \sin(\Omega t), \end{aligned} \quad (9)$$

$$\begin{aligned} \ddot{Q}_y = & -\cos(2\pi Q_x) \sin(2\pi Q_y) \\ & - 2\kappa Q_y. \end{aligned} \quad (10)$$

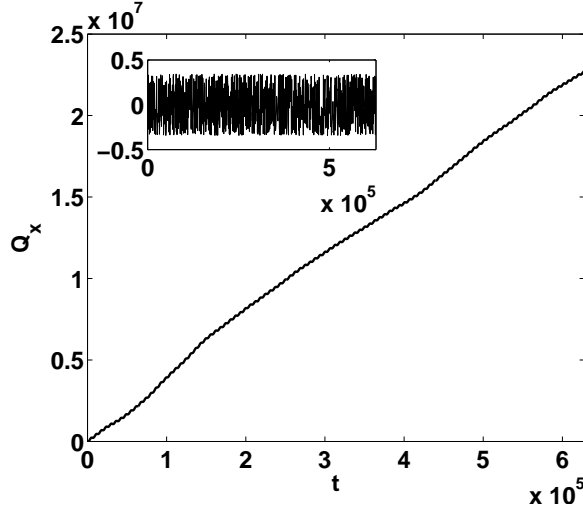


Figure 1: Two units: Time evolution of the variable  $Q_x = (q_1 + q_2)/2$  in the driven case of  $F = 0.05$ . The remaining parameter values are  $\kappa = 0.1$ ,  $\Omega = 10^{-3}$  and  $\Theta_0 = 0$ . The initial conditions are  $q_1(0) = 0.337$ ,  $q_2(0) = 0.347$  and  $p_1(0) = p_2(0) = 0$ . Note the large scale on the ordinate. The inset shows bounded chaotic motions of  $Q_x$  in the undriven case  $F = 0$ .

Evidently, the impact of the external modulation and the linear coupling occurs in separate equations. The coupling between the  $Q_x$  and  $Q_y$  degree of freedom results from parametric modulations of the respective potential force term. Apart from that, equation (9) for the mean value variable  $Q_x$  is equivalent to that of the system of noninteracting particles. Thus, the external modulation may lead to an escape of particles from a potential well initiating ongoing rotational motion of the mean value  $Q_x$  equivalent to the behavior observed in the study of noninteracting particles [17]. Distinct to that the amplitude of the difference variable  $Q_y$  is bounded due to the harmonic restoring force associated with the  $\kappa$ -term.

In our dynamical studies the initial conditions are chosen such that the units are initially contained in a potential well. Furthermore, in the uncoupled case, i.e  $\kappa = 0$ , escape from the potential well induced by the driving force, as discussed in [17], is excluded. To ensure trapping in the driven but uncoupled dynamics the orbits have to lie fairly deep inside the separatrix where there is a large island of stability and the dynamics is still regular.

Likewise, without the *weak* external ac-field and when the units are coupled the chain is supposed to remain trapped in the potential well despite the arising chaotic motion. The latter situation is illustrated in the inset in the top panel of Fig. 1. The simulation time

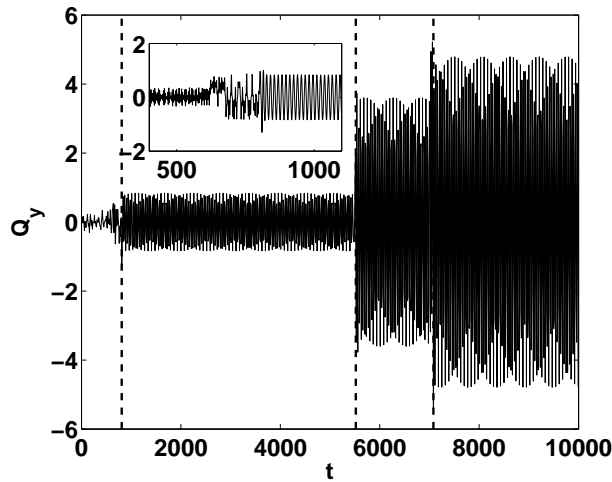


Figure 2: Illustration of the changes in amplitude of the coordinate  $Q_y = (q_1 - q_2)/2$  in the time interval  $0 \leq t \leq 10^4$ . The initial conditions and parameter values are those of the driven case in Fig. 1. Vertical dashed lines mark the instants of time when the amplitude (torus-torus) transitions occur. The inset displays the transition from irregular motion prior to escape to quasi-regular periodic motion after escape.

interval corresponds to a hundred times of the period duration of the external modulation  $T = 2\pi/\Omega$  with  $\Omega = 10^{-3}$ .

However, when the interacting particles are exerted also to the external ac-force escape of the particles from the potential well occurs and results such as those depicted in Fig. 1 are obtained. Interestingly, for a slow modulation with frequency  $\Omega = 10^{-3}$ , after escape from the potential well taking place at  $t \simeq 800$ , there occurs long-range directed transport of the particles equivalent to the behavior reported in [17] for the system of noninteracting particles. Notice the huge distance covered by the particles. As reflected in the evolution of  $Q_y$  (cf. inset in Fig. 2) prior to escape the motion is irregular whereas after escape  $Q_y$  oscillates seemingly regularly indicating capture of the motion near an invariant torus (see further below). That is, the individual coordinates  $q_1$  and  $q_2$  perform oscillations with a mutual phase lag. Moreover, as Fig. 2 reveals the  $Q_y$  quasi-stable oscillations with a certain maximal amplitude are sustained over a specific time interval at the end of which a rather abrupt change of the amplitude occurs and the trajectory comes to lie close to another torus. On a finer scale one recognizes that the transition from one torus to another one is accompanied by a chaotic interlude (see also [23]).

For further illustration of the dynamics in the five dimensional phase space spanned by the variables  $(P_x, P_y, Q_x, Q_y, \Theta = \Omega t)$  we utilize Poincaré-plots where the cross-section is determined as follows:

$$\Sigma := \{(P_x, Q_x, \Theta) | Q_y = 0, P_y > 0\} . \quad (11)$$

We represent in Fig. 3 the Poincaré-plot in the  $\Theta - P_x$ -plane corresponding to the dynamics of the adiabatically driven system shown in Fig. 1. The momentum  $P_x$  experiences vast alterations in dependence of the value of the phase  $\Theta$ . To be precise, starting from an non-inclined potential, i.e.  $\Theta_0 = 0$ , the force term  $F(t) = F \sin(\Omega t)$  produces a negative inclination of the potential in the interval  $0 < t \leq \pi/\Omega$ . It happens that at some instant of time  $t_{escape} \leq \pi/(2\Omega)$  the coordinate  $Q_x$  overcomes the potential barrier and escapes from the well. Hence there remains a time interval  $[t_{escape}, \pi/\Omega]$  during which the particle still experiences a force with positive sign which accelerates further the motion in the right direction towards increasing values of  $Q_x$ . For times  $\pi/\Omega < t \leq 2\pi/\Omega$  the force acts in the opposite direction. In particular for  $2\pi/\Omega - t_{escape} \leq t \leq 2\pi/\Omega$  the momentum evolves with its sign reversed compared to the previous acceleration period. With this we can estimate the gain in momentum as follows

$$\begin{aligned} \Delta P_x &= \left( \int_{t_{escape}}^{\pi/\Omega} + \int_{2\pi/\Omega - t_{escape}}^{2\pi/\Omega} \right) d\tau \dot{p} \\ &= \left( \int_{t_{escape}}^{\pi/\Omega} + \int_{2\pi/\Omega - t_{escape}}^{2\pi/\Omega} \right) \\ &\quad \times d\tau [-\sin(2\pi Q_x) \cos(2\pi Q_y) + F \sin(\Omega \tau)] . \end{aligned} \quad (12)$$

For small  $\Omega$  the rapidly oscillating part connected with the first term in the integral averages to zero and we find

$$\Delta P_x = 2 \frac{F}{\Omega} \cos(\Omega t_{escape}) . \quad (13)$$

In general, the smaller  $\Omega$  the higher is the gain in momentum. In principle, for a sufficiently small frequency  $\Omega$  the gain can become arbitrarily large (see also [16]). With the choice  $\Theta_0 = \pi$  for the initial phase of the modulation term the direction of the motion can be reversed. In contrast in the undriven case, where the chaotic motion remains trapped in the potential well (as depicted in the inset of Fig. 1), the momentum variable  $P_x$  accordingly stays within the range determined by the extension in momentum direction of the corresponding separatrix loop (not shown).

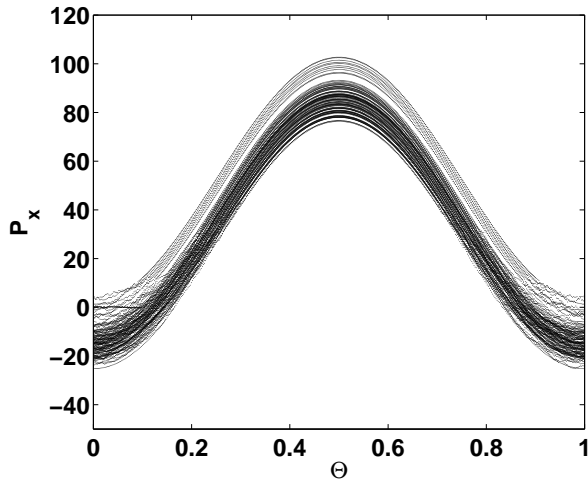


Figure 3: Poincaré-plot presented in the  $\Theta - P_x$ -plane. The phase variable  $\Theta$  is given  $\text{mod } (2\pi)$ . The parameter values are  $\kappa = 0.1$ ,  $F = 0.05$ ,  $\Omega = 10^{-3}$  and  $\Theta_0 = 0$ .

The temporal behavior of the partial energies  $E_{1,2} = 0.5p_{1,2}^2 + U_0(q_{1,2})$  is illustrated in Fig. 4. For times  $t \leq 800$  the energies do not considerably exceed the separatrix level,  $E_s = 1/(2\pi)$ , which is marked by the dashed horizontal line. Escape from the potential well takes place at  $t_{\text{escape}} \simeq 800$  giving rise to significant increase of the energies reaching a maximal value being equivalent to  $10^5$  times the energetic barrier height  $\Delta E = 1/\pi$ . The two partial energies, and likewise the coordinates and momenta, perform oscillations with mutual phase lag (see also further above). The inset displays the temporal evolution of the energies over one period of the external modulation term where one cannot discern the individual trajectories on this scale.

In terms of phase space structure we recall that in the case of individual particles (i.e.  $\kappa = 0$ ) in the corresponding adiabatically driven system of one-and-a-half degree of freedom there arises a broad stochastic layer from the connection of various zones of instability due to resonance overlap. Non-contractible KAM tori confine the stochastic layer from below and above and form impenetrable barriers for motion in phase space. For the weakly nonintegrable system the chaotic sea contains still islands of regular motion. Provided these islands possess non-zero winding numbers orbits with initial condition inside such an island facilitate transport. Moreover, the motion around these islands is characterized by the stickiness to them that can lead to trapping of the trajectory for a long time [24],[25]. This is due the intricate structure of the stochastic layer where close to resonances at the boundary

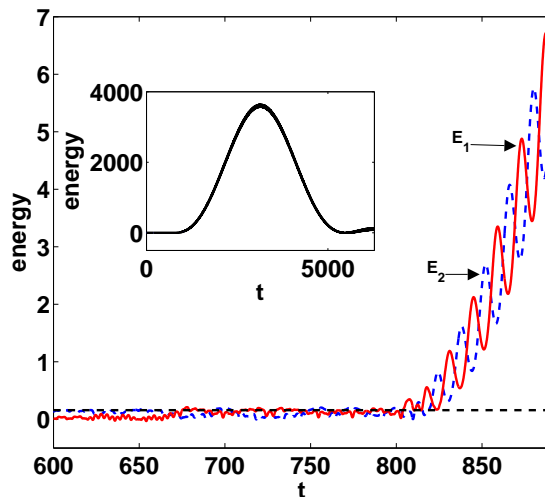


Figure 4: Temporal behavior of the partial energies in the driven system of two coupled units around the trapping-detrapping transition. Initial conditions and parameter values as in Fig. 1. The dashed horizontal line marks the separatrix energy  $E_s = 1/(2\pi)$ . The inset illustrates the drastic growth of the energies during a period  $T = 2\pi/\Omega$ .

between regular and chaotic regions there exists a hierarchy of smaller and smaller islands and surrounding cantori. The latter can severely restrict the transport in phase space and thus effectively partition the chaotic layer [25]. It seems that the cantori are the less leaky the smaller the modulation frequency  $\Omega$ . Hence they form almost impenetrable barriers that confine trajectories for a very long but *transient* period. One should remark that eventually this transient period of directed motion terminates because the trajectory escapes through one of the holes in the cantori and accesses other regions of the chaotic layer. Therefore the motion does not necessarily proceed unidirectionally and unless the trajectory gets captured by ballistic channels [18] it itinerates within the chaotic layer going along with changes of the direction of motion.

For nonlinear Hamiltonian systems with  $N \geq 2$  degrees of freedom only a few numerical results addressing the existence of an enhanced trapping regime are known [26]. It is supposed that the role played by cantori in driven systems with  $N = 1$  is played by families of  $N$ -dimensional tori, constituting partial barriers in the  $2N$ -dimensional phase space, where the chaotic trajectory can stick to [26]. On the other hand Arnold diffusion is possible and hence in principle a chaotic trajectory, wandering along the entire stochastic layer,

can explore the whole phase space [19],[21]. However, due to stickiness to higher dimensional invariant tori Arnold diffusion can be suppressed so that certain stochastic regions are distinguished in which the trajectories become trapped for longer times [26].

In fact, our findings imply that in the driven two-particle system the motion takes place in ballistic channels [18] associated with stickiness to two-dimensional invariant tori (cf. Fig. 3). Notably, in the two particle system the mean value  $Q_x$  evolves in the same manner as the single coordinate in the individual particle counterpart, viz. it exhibits effective growth. We performed numerical studies for a large number of initial conditions contained in a potential well and found behavior equivalent to that illustrated in Figs. 1-4. The slight alterations in the slope of the graph of  $Q_x$  in Fig. 1 can be explained with the fact that after a while the trajectory can get released from a torus region to be captured in another one where it again sticks to for a certain time. As noted above this release and capture behavior is manifested in the changes of the maximal amplitudes of the oscillations of the variable  $Q_y$  (see Figs. 1 and 2).

Alternatively to the geometrical phase space interpretation of the occurrence of directed motion supported by ballistic channels it is worthwhile to mention that from an analytical point of view the so called gating mechanism can be responsible for the initiation of the motion [27]. This mechanism is based on an interplay between an ac driving force and an ac parametric force causing dynamical symmetry breaking that may lead to directed motion. In fact Eq. (9) for the mean value variable  $Q_x$  exhibits both features, i.e. it contains an ac driving term as well as the ac parametric force suggesting relevance of the gating mechanism.

#### IV. EXTENDED COUPLED OSCILLATOR SYSTEMS

Finally we consider extended coupled oscillator chains where we aim to demonstrate that it is also possible to generate directed motion with adiabatic periodic modulations of the slope of the spatially periodic potential.

Concerning the initial conditions we proceed as follows: Initially an amount of energy  $E_n = 0.5p_n^2 + U_0(q_n) < \Delta E$  is applied per unit such that the whole chain is elongated homogeneously along a fixed position  $\tilde{q}_0$  near the bottom of the well. Then, the position and/or momenta of all units are randomized. The random position values are chosen from a

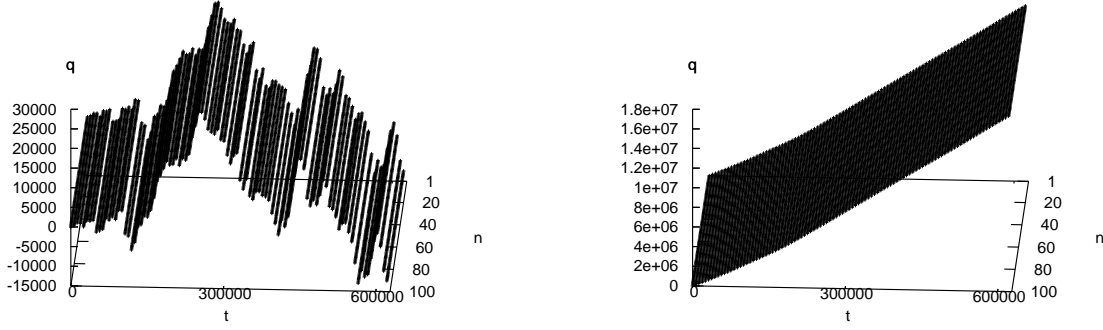


Figure 5: Typical spatio-temporal evolution of the coordinates  $q_n(t)$  for a chain of 100 coupled particles. Initially the coordinates are uniformly distributed within the interval  $|q_n(0) - \tilde{q}_0| < \Delta q$  with an average  $\tilde{q}_0 = -0.35$  and width  $\Delta q = 0.01$  and zero momenta, i.e.  $p_n(0) = \tilde{p}_0 = \Delta p = 0$ . The coupling strength is  $\kappa = 0.3$  and the remaining parameter values are  $F = 0.05$ ,  $\Omega = 10^{-3}$  and  $\Theta_0 = 0$ . Top (Bottom) panel:  $\Omega = 10^{-1}$  ( $\Omega = 10^{-3}$ ).

bounded interval  $|q_n(0) - \tilde{q}_0| \leq \Delta q$  and, likewise, the random initial momenta,  $|p_n(0) - \tilde{p}_0| \leq \Delta p$ . The whole chain is thus initialized close to an almost homogeneous state, but yet sufficiently displaced ( $\Delta q \neq 0$ ) in order to generate nonvanishing interactions, entailing the exchange of energy among the coupled units.

First we note that in the case without external modulation of the slope there occurs the formation of a pattern of localized states due to modulational instability (not shown here). Due the irregular dynamics it happens that occasionally a unit overcomes the potential barrier but no coordinated motion of the chain results.

Remarkably, applying the adiabatic modulation the entire chain not only escapes from the potential well but manages also to travel freely and unidirectionally over a giant distance as seen from the spatio-temporal evolution of  $q_n(t)$  in Fig. 5. In comparison for faster modulations no directed motion of the chain is obtained. The chain consists of 100 coupled oscillators and open boundary conditions are imposed. The profile of the chain continuously undergoes changes with ensuing deviations from a flat state. Nevertheless the intriguing feature of transients of extremely long-range transport of the chain is provided by collective motion which is also reflected in the temporal behavior of the mean value of the coordinate  $\langle q \rangle = \langle \frac{1}{N} \sum_{n=1}^N q_n \rangle$  shown in Fig. 6. Ensemble averages, denoted by  $\langle \rangle$  were performed

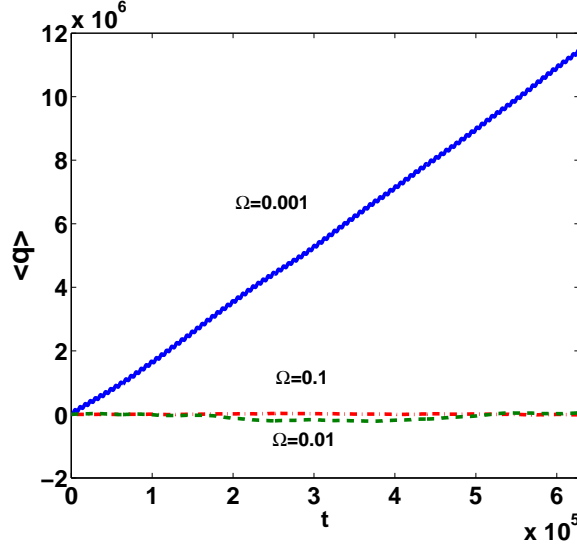


Figure 6: Temporal evolution of the mean value  $\langle q \rangle$  for systems driven with different driving frequencies as indicated in the plot. Ensemble averages, denoted by  $\langle \rangle$  were performed over 100 realizations of trajectories whose initial condition lie in the range given in Fig. 5. The parameter values are those given in Fig. 5.

over 100 realizations of trajectories whose initial condition lie in the range given in the caption of Fig. 5. Note that while for driving with  $\Omega \lesssim 10^{-2}$  on average no directed motion results for sufficiently slow modulations (here illustrated for  $\Omega = 10^{-3}$ ) a large directed current is observed. Conclusively, concerning transport the many particle system exhibits features equivalent to that observed in the system of individual particles [16],[17].

It is interesting to relate our findings to those obtained for the continuum case of Eq. (5), viz. a perturbed sine-Gordon equation that is driven by a mono-frequency periodic external field destroying integrability [28]. There also the emergence of a directed current for driven solitons results for suitable parameter values of the external field. In comparison we note that in the 1D continuum case a localized wave-like excitation in the form of a soliton moves along the spatial coordinate whereas in our present study of the discrete counterpart the entire chain is coherently transported over the wells of the periodic potential landscape.

## V. SUMMARY

To summarize, we found that in the time-dependent driven Hamiltonian dynamics of a chain of linearly interacting particles evolving in a symmetric, spatially periodic potential directed transport can be induced. That is, when the slope of the potential is time-periodically varied due to a slowly oscillating external modulation field the transport proceeds unidirectionally in a long-lasting transient period such that the chain covers huge distances. The behavior of the mean value of the coordinates of extended chains is strikingly similar to that found in the system of noninteracting particles. Thus, the collective directed motion of the numerous microscopic degrees of freedom is manifested at a collective level in the evolution of the mean value of  $q$ . We recall that for the system of individual particles transporting island structures corresponding to ballistic channels present in mixed phase space are responsible for transients of directed long-range transport. Our findings for the many particle system imply that ballistic channels exist in a high dimensional phase space too and support the collective motion of the coupled oscillators of a chain. For systems involving many degrees of freedom the issue of stickiness of invariant tori in high-dimensional phase space needs to be explored in more detail (an analysis which goes beyond the scope of the current article).

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